Collision Detection in Aspect and Scale Bounded Polyhedra

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Abstract

Let $\mathcal{S}$ be a family of n polyhedral objects in d dimensions, with aspect ratio bound α and scale factor bound σ. Let $K_o(\mathcal{S})$ denote the number of object-pairs in $\mathcal{S}$ with nonempty intersection, and let $K_b(\mathcal{S})$ denote the number of pairs whose enclosing balls intersect. We investigate the worst-case behavior of the following ratio:

$$\rho(\mathcal{S}) = \frac{K_b(\mathcal{S})}{n + K_o(\mathcal{S})}.$$

We establish almost-tight asymptotic bounds: $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma}\log^2\sigma)$, and $\rho(\mathcal{S}) = \Omega(\alpha\sqrt{\sigma})$. The important conclusion is that the ratio is independent of n, and if $\mathcal{S}$ has bounded aspect ratio and scale factor, the number of enclosing ball-pair intersections is about the same as the number of object-pair intersections.

Our theorem implies the following two results. First, it lends strong theoretical support to a simple and practical heuristic for collision detection (the bounding box method), used in application domains such as computer graphics and robotics, where the objects typically have constant aspect ratio and scale factor. Second, it yields an output-sensitive algorithm for reporting all intersecting pairs in a set of n *convex* polyhedra with constant α and σ. Our algorithm runs in time $O(n\log^{d-1} n + K_o \log^{d-1} n)$, for $d = 2, 3$, where K_o is the number of intersecting object pairs. This is significantly better than the bounds achieved by the previous algorithms, which make no assumptions about the aspect and scale factors.

1 Introduction

Interference and collision detection are fundamental problems in a variety of application domains, such as physically-based modeling, robotics, animation, computer-aided design, manufacturing, and computer-simulated environments. CAD/CAM systems, for instance, use collision detection for clearance verification in an assembly [7, 12], robot systems use collision detection for path planning [15] and, in computer graphics, collision detection works in conjunction with collision response to make animation appear more realistic and believable [2, 14, 17].

Typically, a collision-detection algorithm takes as input a collection $\mathcal{S}$ of n d-dimensional objects, and produces as output the pairs of objects that intersect. A high performance system can involve a large number of complicated objects, and demand accurate collision detection at real time rates. Klosowski et al. [14] state that technologies such as haptic force-feedback may require over 1000 collision queries per second. Most practical systems for collision detection divide their work into two phases, which we call *broad phase* and *narrow phase*. In the broad phase, the algorithm typically uses a conservative but simpler approximation of each object to find pairs whose approximations intersect. The narrow phase then performs detailed intersection tests on pairs of objects found by the broad phase. Several different approximations for geometric objects are possible: axis-aligned bounding box, minimum enclosing sphere, oriented box, or discrete orientation polytope [2, 13, 14]. The exact choice of an approximating shape is not critical for our research, but for the sake of concreteness, let us consider the axis-aligned bounding boxes. See Figure 1(a) for an example. We can write the two-phase collision detection procedure as follows:

- [BROAD PHASE.] Find all pairs of intersecting bounding boxes.
- [NARROW PHASE.] For each intersecting pair found by the broad phase, perform a detailed intersection test on the corresponding objects.

The broad and narrow phases have distinct characteristics, and often have been treated as separate pieces of the collision detection problem in the research community. Specifically, the narrow phase considers the

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problem of contact area determination: find precise intersection between facets of the two polyhedra. Thus, the performance of a narrow phase algorithm *does not* depend on n, the number of objects in the set, but rather on the complexity of each object. The broad phase looks at all n objects in the scene, and reports the pairs that potentially intersect. An efficient broad phase algorithm must avoid looking at all $\binom{n}{2}$ pairs. Ideally, one would want an *output-sensitive* algorithm, whose running time is proportional to the number of colliding pairs.

Our research is concerned with identifying some natural measures that would explain the practical efficiency of the bounding box heuristic, despite its poor worst-case bound. We show that two parameters, *aspect ratio* α and *scale factor* σ, play a critical role in the analysis of the broad phase. Specifically, we prove that the number of bounding box intersections does not exceed about $\alpha\sqrt{\sigma}$ times the number of object-pair intersections—the parameters α and σ typically have small constant values in practice.

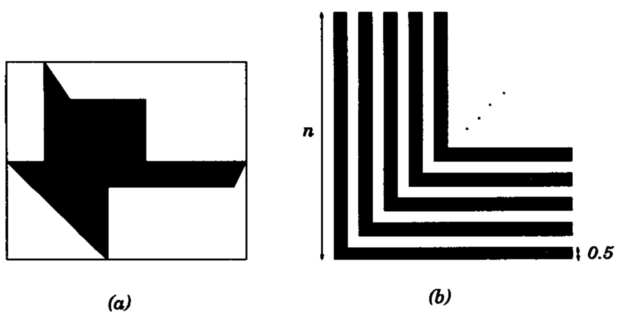

Figure 1: (a) A polygonal object and its axis-aligned bounding box. (b) An example with $K_b = \Omega(n^2)$ and $K_o = O(1)$.

The aspect ratio measures the *elongatedness* of an object. It is often defined as the ratio between the volumes of the smallest ball enclosing the object and the largest ball contained in the object. We will find it convenient to use the volumes of L_∞-norm balls in the d-space.[1] Given a solid polyhedral object P in d-space, let $b(P)$ denote the smallest L_∞ ball containing P, and let $c(P)$ denote the largest L_∞ ball contained in P. The aspect ratio of P is defined as

$$\alpha(P) = \frac{\mathrm{vol}(b(P))}{\mathrm{vol}(c(P))},$$

where vol(P) denotes the d-dimensional volume of P. We will call $b(P)$ the *enclosing box*, and $c(P)$ the *core* of P. Thus, the aspect ratio measures the volume of the enclosing box relative to the core.

Let $\mathcal{S} = \{P_1, P_2, \ldots, P_n\}$ be a set of polyhedral objects in d-space, with aspect ratio bound α, meaning that $\alpha(P_i) \leq \alpha$, for $i = 1, 2, \ldots, n$. We say that family $\mathcal{S}$ has *scale factor* σ if, for all $1 \leq i, j \leq n$,

$$\frac{\mathrm{vol}(b(P_i))}{\mathrm{vol}(b(P_j))} \leq \sigma.$$

1.1 Statement of Results.

Let K_o be the number of object pairs in $\mathcal{S}$ with nonempty intersection, and let K_b denote the number of object pairs whose enclosing boxes intersect.[2] (The set $\mathcal{S}$ will be clear from the context throughout, so we omit its explicit mention from our notation in the rest of the paper.) We are interested in the following ratio:

$$\rho(\mathcal{S}) = \frac{K_b}{n + K_o}.$$

The denominator represents the work done by an *ideal* broad phase algorithm, and so the ratio can be seen the relative performance measure of the bounding box heuristic. Ideally, one would like this ratio to be a small constant. Unfortunately, the pathological case of Figure 1(b) shows that without any assumptions on α and σ, we can have $\rho(\mathcal{S}) = \Omega(n)$. However, if we include aspect ratio and scale factors in the analysis, we can prove the following theorem, which is the main result of our paper.

THEOREM 1.1. *Let $\mathcal{S}$ be a set of n polyhedral objects in d dimensions, with aspect bound α and scale factor σ, where d is a constant. Then, $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma}\log^2\sigma)$. Asymptotically, this bound is almost tight, as we can show a family $\mathcal{S}$ achieving $\rho(\mathcal{S}) = \Omega(\alpha\sqrt{\sigma})$.*

There are two main implications of this theorem. First, we get an output-sensitive algorithm for reporting all pairs of intersecting objects in a set of n *convex polyhedra* in two or three dimensions. If there are K_o intersecting pairs, then the bounding box algorithm reports them in time $O(n \log^{d-1} n + K_b \log^{d-1} m)$ time, for $d = 2, 3$, where $K_b = O(K_o \alpha\sqrt{\sigma}\log^2\sigma)$, and m is the maximum number of vertices in a polyhedron. (We assume that polyhedra have been preprocessed in linear time for efficient pairwise intersection detection [4].) Without the aspect and scale bounds, we are not aware

[1] In two dimensions, for instance, the L_∞ ball of radius r and center o is the axis-aligned square of side length $2r$, with center o. The choice of the norm affects only the small dimension-dependent constant factors, and our results apply also to L_2 balls or other commonly used norms with small changes in the constant.

[2] Notice that the L_∞ ball is a more conservative estimate than the axis-aligned bounding box and so K_b is an upper bound on the number of bounding box intersections.

of any output-sensitive algorithm for this problem in three dimensions. Even in two dimensions, the best algorithm for finding all intersecting pairs in a set of n convex polygons takes $O(n^{4/3} + K_o)$ time [9]. If the aspect and scale factors are constants, a common case in applications, our algorithm runs in time $O(n \log^{d-1} n + K_o \log^{d-1} m)$, for $d = 2, 3$, which is close to optimal.

Second, our result gives a theoretical explanation for the observed efficiency of the bounding box heuristic for collision detection in applications mentioned earlier. In these applications, object families typically have small aspect ratio and scale factor, and so our theorem implies that the bounding box heuristic is asymptotically optimal. This result does not depend on any convexity assumption about the polyhedra: it simply states that the number of pairs for which the narrow phase computation is performed is at most ρ times the best possible number; in practice, ρ is typically a small constant. Thus, pathological cases like the ones in Figure 1(b) are unusual since they require that $\alpha\sqrt{\sigma} = \Omega(n)$. Our theorem also shows that the dependence on aspect ratio is more severe than the scale factor, and thus it may be worthwhile to decompose complex objects into smaller pieces to improve the aspect bound.

1.2 Previous Work.

The fundamental nature and broad applications of collision detection have made it an active topic of research. The problems are quite hard both in broad phase as well as in narrow phase, and *provably* efficient algorithms are known only for highly specialized cases.

Let us first consider the narrow phase. If the objects are *convex polyhedra,* then a method due to Dobkin and Kirkpatrick [4] can decide whether two objects intersect in $O(\log^{d-1} m)$ time, where m is the total number of edges in the two polyhedra, and $d \leq 3$ is the dimension. The polyhedra require a linear time preprocessing phase. Using this preprocessing, one can also compute an explicit representation of the intersection of two convex polyhedra in time $O(m)$, as shown by Chazelle [1]. If only one of the objects in the pair is convex, then intersection detection can be performed in time $O(m \log m)$ [3]. The problem is more difficult when both polyhedra are nonconvex, and only recently has a subquadratic time algorithm been discovered for deciding if two nonconvex polyhedra intersect [19]. This algorithm takes $O(m^{8/5+\epsilon})$ time to determine the first collision between two polyhedra, one of which is stationary and the other is translating. These theoretical algorithms employ many novel ideas and sophisticated data structures, but they are deemed too complicated and slow to be practical. Instead a variety of heuristic methods have been developed that tend to work well in practice [8, 14]. These methods use hierarchies of bounding volumes and tree-descent schemes to determine intersections.

The broad phase problem can be formulated as follows: Given n objects in d-space, find all intersecting object pairs. Provably efficient algorithms are known only for highly specialized objects, such as axis-aligned rectangular boxes. Specifically, one can find k intersecting pairs of axis-aligned rectangular boxes in d-space in $O(n \log^{d-1} n + k)$ time and $O(n \log^{d-2} n)$ space [6, 16]. No efficient algorithm is known for finding all intersecting pairs even in a set of n *convex polyhedra* in 3D. If the objects are convex polygons in two dimensions, then a recent algorithm of Gupta et al. [9] can report the intersecting pairs in time $O(n^{4/3} + k)$. A variety of heuristic methods are used in practice [2, 13], whose performance is typically analyzed empirically. The "sweep-and-prune" algorithm implemented in the I-COLLIDE package of Cohen et al. [2] currently appears to be method of choice. It falls under the generic bounding box class of heuristics, and as such our analysis applies to it.

1.3 Organization.

Our proof for the upper bound on $\rho(\mathcal{S})$ consists of three steps. We first consider the case of arbitrary α but fixed σ. Next, we allow both α and σ to be arbitrary but assume that there are only two kinds of objects: one with box sizes α and the other with box sizes $\alpha\sigma$ (the two extreme ends of the scale factor). Finally, we handle the general case, where objects can have any box size in the range $[\alpha, \alpha\sigma]$. We first detail our proof for two dimensions, and then sketch how to extend it to arbitrary dimensions.

2 Arbitrary Aspect Ratio but Fixed Scale

We start by assuming that the set $\mathcal{S}$ has scale factor one, that is, $\sigma = 1$; the aspect ratio bound α can be arbitrary. (Any constant bound for σ will work for our proof; we assume one for convenience. The most straightforward way to enforce this scale bound is to make every object's enclosing box to be the same size.) We will show that in this case $\rho(\mathcal{S}) = O(\alpha)$. We describe our proof in two dimensions; the extension to higher dimensions is quite straightforward, and is sketched in Section 5.

Without loss of generality, let us assume that each object P in $\mathcal{S}$ has $\mathrm{vol}(c(P)) \geq 1$, and $\mathrm{vol}(b(P)) \leq \alpha$. Recall that a L_∞ box of volume α in two dimensions is a square of side length $\sqrt{\alpha}$. We call this a *size α box*. Consider a tiling of the plane by size α boxes that covers the portion of the plane occupied by the bounding boxes of the objects, namely, $\bigcup b(P_i)$. See Figure 2. We will consider each box semi-open, so that the boundary shared by two boxes belongs to the one on the left, or above. Thus, each point of the plane belongs to at most one box.

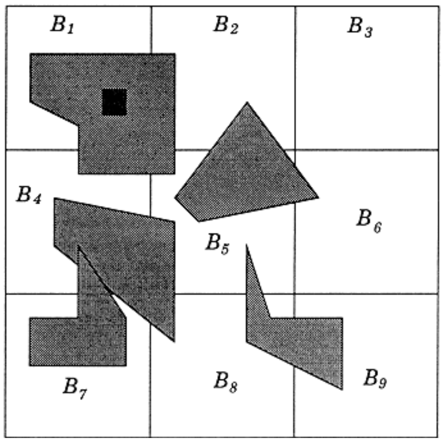

Figure 2: Tiling of the plane by boxes of size α. The unit size core for the object in B_1 is also shown.

We assume an underlying unit lattice in the plane, and assign each object P to the (unique) *lexicographically smallest lattice point* contained in P. (Such a point exists because the core is closed and has volume at least one.) Let $m(q)$ be the number of objects assigned to a lattice point q, and let M_i denote the total number of objects assigned to the lattice points contained in a box B_i. That is,

$$M_i = \sum_{q \in B_i} m(q),$$

where $q \in B_i$ means that the lattice point q lies in the box B_i. Since the boxes in the tiling are disjoint, we have the equality $\sum_i M_i = n$. We will derive the bounds on K_b and K_o in terms of M_i.

LEMMA 2.1. *Given a set of objects $\mathcal{S}$ with aspect bound α and scale bound $\sigma = 1$, let $B_1, B_2, \ldots, B_p$ denote a tiling by size α boxes as defined above, and let M_i denote the total number of objects assigned to lattice points in B_i, for $i = 1, 2, \ldots, p$. Then,*

$$K_b \leq 25 \sum_{i=1}^{p} M_i^2.$$

Proof. Consider an object P assigned to B_i, and let P_j be another object whose box intersects $b(P)$. Suppose P_j is assigned to the box B_j. Since $b(P) \cap b(P_j) \neq \emptyset$, the L_∞ norm distance between the boxes B_i and B_j is at most $2\sqrt{\alpha}$. This means that B_j is among the 24 boxes that lie within $2\sqrt{\alpha}$ wide corridor around B_i.

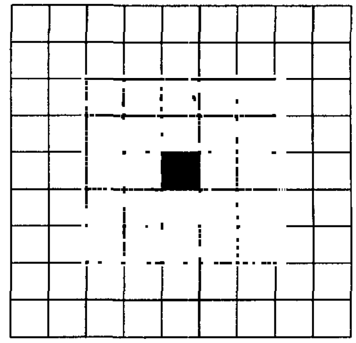

Figure 3: A box, shown in dark at the center, and its 24 neighbors.

Suppose that the boxes are labeled $B_1, B_2, \ldots, B_p$ in the row-major order—top to bottom, left to right in each row. Assume that the number of columns in the box tiling is k. Then, the preceding discussion shows that if the boxes of objects P_i and P_j intersect and these objects are assigned to boxes B_i and B_j, then we must have

$$j = i + ck + d,$$

where $c, d \in \{-2, -1, 0, 1, 2\}$. (The box B_j can be at most two rows and two columns away from B_i. For instance, the box preceding two rows and two columns from B_i is B_{i-2k-2}.) See Figure 3. The number of box pair intersections contributed by B_i and B_j is clearly no more than $M_i M_j$. Thus, the total number of such intersections is bounded by

$$\sum_{i=1}^{p} \sum_{j=i+ck+d} M_i M_j,$$

where $c, d \in \{-2, -1, 0, 1, 2\}$. Recalling that $x_1 x_2 \leq \frac{1}{2}(x_1^2 + x_2^2)$, for reals x_1, x_2, we can bound the intersection count by

$$\sum_{i=1}^{p} \sum_{j=i+ck+d} \frac{1}{2}(M_i^2 + M_j^2).$$

There are 5 possible values for c and d each, and so altogether 25 values for j for each i. Since each index can appear once as the i and once as the j, we get that the maximum number of intersections is at most

$$25 \sum_{i=1}^{p} M_i^2.$$

This completes the proof of the lemma. □

Next, we establish a lower bound on the number of intersecting object pairs. We will need the following elementary fact.

LEMMA 2.2. *Consider non-negative numbers $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$. Then,*

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \max_{1 \leq i \leq k} \frac{a_i}{b_i}.$$

Proof. Let m denote the index for which the ratio a_i/b_i is maximized. Since $b_i(a_m/b_m) \geq a_i$, summing it over all i, we get

$$\frac{a_m}{b_m} \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} a_i.$$

Dividing both sides by $\sum_{i=1}^{n} b_i$ completes the proof of the lemma. □

Let us now focus on objects assigned to a box B_i in our tiling. If L_i is the number of intersecting pairs among objects assigned to B_i, then we have the following:

$$\begin{aligned}
\rho(\mathcal{S}) &= \frac{K_b}{n + K_o} \\
&\leq \frac{25 \sum_{i=1}^{p} M_i^2}{n + \sum_{i=1}^{p} L_i} \\
&\leq \frac{25(M_1^2 + M_2^2 + \cdots + M_p^2)}{(M_1 + L_1) + (M_2 + L_2) + \cdots + (M_p + L_p)} \\
&\leq \max_{1 \leq i \leq p} \frac{25 M_i^2}{(M_i + L_i)},
\end{aligned}$$

where the second to last inequality follows from the fact that $\sum_i M_i = n$, and the last inequality follows from the preceding lemma. We will establish an upper bound on the right hand side of this inequality by proving a lower bound on the denominator term.

Fix a box B_i in the following discussion, where $1 \leq i \leq p$, and consider a lattice point q in it. Since $m(q)$ objects have q in common, at least $\binom{m(q)}{2}$ object pair intersections are contributed by the objects assigned to q. (Observe that each object is assigned to a unique lattice point, and so we count each intersection at most once.) Thus, the total number of pairwise intersections L_i among objects assigned to B_i is at least

$$\sum_{q \in B_i} \binom{m(q)}{2}.$$

We will show that the ratio $25M_i^2/(M_i + L_i)$ never exceeds $c\alpha$, where c is an absolute constant. Considering M_i fixed, this ratio is maximized when L_i is minimized.

LEMMA 2.3. *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers that sum to z. The minimum value of $\sum_{i=1}^{n} \binom{x_i}{2}$ is $z(z-n)/2n$, which is achieved when $x_i = z/n$, for $i = 1, 2, \ldots, n$.*

Proof. We observe the following equalities:

$$\sum_{i=1}^{n} \binom{x_i}{2} = \sum_{i=1}^{n} \frac{x_i(x_i - 1)}{2} = \frac{1}{2} \left(\sum_{i=1}^{n} x_i^2 - z \right)$$

Thus, $\sum_{i=1}^{n} \binom{x_i}{2}$ is minimized when $\sum_{i=1}^{n} \frac{x_i^2}{2}$ is minimized. Using Cauchy's Inequality [10], the latter is minimized when $x_i = z/n$. The lemma follows. □

Since no square box of size α can have more than $2\lceil \alpha \rceil$ lattice points in it, we get a lower bound on L_i by setting $m(q) = \frac{M_i}{2\lceil \alpha \rceil}$, for all q. Thus,

$$L_i \geq \frac{1}{2} M_i \left(\frac{M_i}{2\lceil \alpha \rceil} - 1 \right).$$

LEMMA 2.4. $\rho(\mathcal{S}) = O(\alpha)$.

Proof. Using the bound for L_i above, we have

$$\rho(\mathcal{S}) \leq \frac{25 M_i^2}{\frac{1}{2} M_i \left(\frac{M_i}{2\lceil \alpha \rceil} - 1 \right) + M_i} \leq 100 \lceil \alpha \rceil.$$

This completes the proof. □

THEOREM 2.1. *Let $\mathcal{S}$ be a set of n objects in the plane, with aspect bound α and scale bound $\sigma = 1$. Then, $\rho(\mathcal{S}) = O(\alpha)$.*

3 Objects of two Fixed Sizes

In this section, we generalize the result of the previous section to the case where objects come from the two extreme ends of the scale: their box size is either α or $\alpha\sigma$. To simplify our analysis, we will assume that $\alpha = 4^a$ and $\sigma = 4^b$ for some integers $a, b > 0$. (Otherwise, just use the next nearest powers of 4 as upper bounds for α, σ. In d dimensions, α and σ are assumed to be integral powers of 2^d.)

Let us call an object *large* if its enclosing box has size $\alpha\sigma$, and *small* otherwise. Clearly, there are only three kinds of intersections: large-large, small-small, and large-small. Let K_b^l, K_b^s and K_b^{sl}, respectively, count these intersections for the enclosing boxes. So, for example, K_b^{sl} is number of pairs consisting of one large and one small object whose boxes intersect. Similarly, define the terms K_o^l, K_o^s and K_o^{sl} for object pair intersections. The ratio bound can now be restated as

$$
\begin{aligned}
(3.1) \qquad \rho(\mathcal{S}) &= \frac{K_b^l + K_b^s + K_b^{sl}}{K_o^l + K_o^s + K_o^{sl} + n} \\
&\leq 3 \max\left\{ \frac{K_b^l}{K'}, \frac{K_b^s}{K'}, \frac{K_b^{sl}}{K'} \right\}
\end{aligned}
$$

where $K' = K_o^l + K_o^s + K_o^{sl} + n$. We know from the result of the previous section that $\frac{K_b^l}{K'}, \frac{K_b^s}{K'} \leq c\alpha$, for some constant c. So, we only need to establish a bound on the third ratio, $\frac{K_b^{sl}}{K_o^l + K_o^s + K_o^{sl} + n}$, which we do as follows.

Let us again tile the plane with boxes of volume $\alpha\sigma$. Call these boxes $B_1, B_2, \ldots, B_p$. Underlying this tiling are two grids: a *level σ grid*, which divides the boxes into cells of size σ, and a *level* 1 *grid*, which divides the boxes into cells of size 1. The level σ grid has vertices at coordinates $(i\sqrt{\sigma}, j\sqrt{\sigma})$, while the finer grid has vertices at coordinates (i, j), for integers i, j. The level σ grid is used to reason about large objects, while the level 1 grid is used for small objects. We will mimic the proof of the previous section, and assign objects of each class to an appropriate box. In order to do that, we need to define subboxes of size α within each size $\alpha\sigma$ original box.

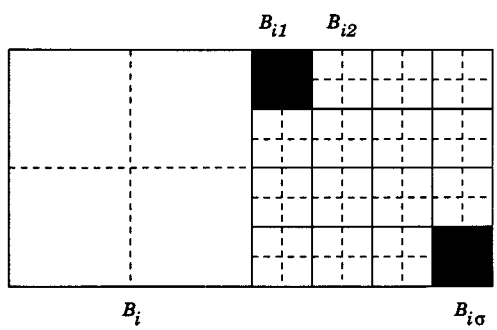

Figure 4: The box on the left shows large grid, and the one on the right shows small grid as well as the subboxes. In this figure, $\alpha = 4$ and $\sigma = 16$.

Consider a large box B_i. The level σ grid partitions B_i into α boxes of volume σ each. Next, we also partition B_i into σ subboxes, each of volume α. Since $\alpha = 4^a$ and $\sigma = 4^b$, for integers $a, b > 0$, these subboxes are perfectly aligned with both the level 1 and level σ grids. (Along a side of B_i, the σ grid has vertices at distance multiples of $\sqrt{\sigma} = 2^b$, while the vertices of the subboxes lie at distance multiples of $\sqrt{\alpha} = 2^a$.) We label the σ subboxes within B_i as $B_{i1}, B_{i2}, \ldots, B_{i\sigma}$, in row major order. Figure 4 illustrates these definitions, by showing two boxes side by side.

Now, each member of the large object set (resp. small object set) contains at least one grid point of the large (resp. small) grid. Just as in the previous section, we assign each object to a unique grid point (say, the one with lexicographically smallest coordinates). Let X_i denote the number of large objects assigned to all the grid points in B_i. Let y_{ij}, for $j = 1, 2, \ldots, \sigma$, denote the number of small objects assigned to the subbox B_{ij}. Define also $Y_i = \sum_{i=1}^{\sigma} y_{ij}$ to be the total number of small objects assigned to level one grid points in B_i.

We estimate an upper bound on K_b^{sl} and a lower bound on K_o^{sl}, in terms of X_i and Y_i. Fix a box B_i. The enclosing box of a large object P_i, assigned to B_i, can intersect the box of a small object P_j, assigned to B_j, only if B_j is one of the 25 neighbors of B_i (including itself) that form the two layers of boxes around B_i. (See Figure 3 again.) Let B_i^m be the box with a maximum number of small objects among the 25 neighbors of B_i, and let Y_i^m be the count of the small objects in B_i^m. That is, $Y_i^m = \max_j\{Y_j \mid B_j \text{ is one of 25 neighbors of } B_i\}$, and B_i^m is the box corresponding to Y_i^m. Then, we have the following upper bound:

$$
K_b^{sl} \leq 25 \sum_{i=1}^{p} X_i Y_i^m.
$$

Next, we estimate lower bounds on the number of object pair intersections. Let L_i denote the number of object pair intersections among the large objects assigned to B_i, and let S_i denote the object pair intersections among the small objects assigned to B_i. Since there are only α grid points for the large objects in B_i, by Lemma 2.3, we have

$$
(3.2) \qquad L_i \geq \frac{1}{2} X_i \left(\frac{X_i}{\alpha} - 1 \right).
$$

Similarly, each of the subboxes B_{ij}, for $j = 1, 2, \ldots, \sigma$, has α grid points of the level 1 grid. Thus, we also have

$$
(3.3) \qquad S_i \geq \sum_{j=1}^{\sigma} \frac{y_{ij}}{2} \left(\frac{y_{ij}}{\alpha} - 1 \right).
$$

In deriving our bound, we will use the conservative estimate of $\sum_{i=1}^{p}(L_i + S_i)$ for K_o; that is, only count the intersections between two large or two small objects. We also use the notation S_i^m for the number of object-pair intersections among the small objects assigned to B_i^m. We have the following inequalities:

$$
\begin{aligned}
\frac{K_b^{sl}}{n + K_o} &\leq \frac{25 \sum_{i=1}^{p} X_i Y_i^m}{n + \sum_{i=1}^{p}(L_i + S_i)} \\
&= \frac{25^2 \sum_{i=1}^{p} X_i Y_i^m}{25 \sum_{i=1}^{p}(X_i + L_i + Y_i + S_i)}
\end{aligned}
$$

$$\leq \frac{25^2 \sum_{i=1}^{p} X_i Y_i^m}{\sum_{i=1}^{p}(X_i + L_i + Y_i^m + S_i^m)}$$
$$\leq \max_{1\leq i\leq p} \frac{25^2 X_i Y_i^m}{X_i + L_i + Y_i^m + S_i^m},$$

where the second inequality follows from the fact that $\sum_{i=1}^{p}(X_i + Y_i) = n$; the third follows from the fact that a particular box B_i^m can contribute the Y_i^m term to at most its 25 neighbors; and the final inequality follows from Lemma 2.2. The remaining step of the proof now is to show that the above inequality is $O(\alpha\sqrt{\sigma})$. First, by summing up the terms in Eqs. (3.2) and (3.3), we observe the following:

$$X_i + L_i + Y_i^m + S_i^m \geq \frac{X_i^2 + \sum_{j=1}^{\sigma}(y_{ij}^m)^2}{2\alpha},$$

where recall that $\sum_{j=1}^{\sigma} y_{ij}^m = Y_i^m$. Thus, we have

$$\begin{aligned}\frac{X_i Y_i^m}{X_i + Y_i^m + L_i + S_i^m} &\leq \frac{2\alpha X_i Y_i^m}{X_i^2 + \sum_{j=1}^{\sigma}(y_{ij}^m)^2} \\ &\leq \frac{2\alpha X_i Y_i^m}{X_i^2 + \sigma(\frac{Y_i^m}{\sigma})^2} \\ &\leq \frac{2\alpha\sigma X_i Y_i^m}{\sigma X_i^2 + (Y_i^m)^2},\end{aligned}$$

where once again Cauchy's inequality is invoked to show that $\sum_{j=1}^{\sigma}(y_{ij}^m)^2 \geq \sigma(\frac{Y_i^m}{\sigma})^2$. It can be easily shown that this ratio is at most $2\alpha\sqrt{\sigma}$, as follows. If $Y_i^m \leq \sqrt{\sigma}X_i$, then we have

$$\frac{2\alpha\sigma X_i Y_i^m}{\sigma X_i^2 + (Y_i^m)^2} \leq \frac{2\alpha\sigma X_i\sqrt{\sigma}X_i}{\sigma X_i^2} \leq 2\alpha\sqrt{\sigma}.$$

Otherwise, $Y_i^m > \sqrt{\sigma}X_i$, and we have

$$\frac{2\alpha\sigma X_i Y_i^m}{\sigma X_i^2 + (Y_i^m)^2} \leq \frac{2\alpha\sigma\frac{Y_i^m}{\sqrt{\sigma}}Y_i^m}{(Y_i^m)^2} \leq 2\alpha\sqrt{\sigma}.$$

This shows that $\frac{K_b^{s\ell}}{n+K_o} = O(\alpha\sqrt{\sigma})$. Combining this with Ineq. (3.2), we get the desired result, which is stated in the following theorem.

THEOREM 3.1. *Suppose $\mathcal{S}$ is a set of n objects in the plane, such that each object has aspect ratio at most α, and the enclosing box of each object has size either α or $\alpha\sigma$. Then, $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma})$.*

4 The General Case

We now are in a position to prove our main theorem. Suppose $\mathcal{S}$ is a set of n polyhedral objects, with aspect ratio bound α and scale factor σ. Recall that for simplicity we assume that both α and σ are powers of four. We partition the set $\mathcal{S}$ into $O(\log\sigma)$ classes, $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$, for $k = \log\sigma$, such that a polyhedron P belongs to class $\mathcal{C}_i$ if $2^i \leq \mathrm{vol}(c(P)) < 2^{i+1}$. (Equivalently, the enclosing boxes of objects in class $\mathcal{C}_i$ have volumes between $\alpha 2^i$ and $\alpha 2^{i+1}$.) Each class behaves like a fixed size family (the case considered in Section 2), and so we have $\rho(\mathcal{C}_i) = O(\alpha)$, for $i = 0, 1, \ldots, \log\sigma$. Any pair of classes behaves like the case considered in Section 3, implying that $\rho(\mathcal{C}_i \cup \mathcal{C}_j) = O(\alpha\sqrt{\sigma})$, for $0 \leq i, j \leq \log\sigma$. We can now formalize this argument to show that $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma}\log^2\sigma)$.

Let K_b^{ij}, for $0 \leq i, j \leq \log\sigma$, denote the number of object pairs (P, P') whose enclosing boxes intersect such that $P \in \mathcal{C}_i$ and $P' \in \mathcal{C}_j$. Similarly, define K_o^{ij}. Then, we have the following:

$$\begin{aligned}\rho(\mathcal{S}) &= \frac{\sum_i\sum_j K_b^{ij}}{\sum_i\sum_j K_o^{ij} + n} \\ &\leq \left(\frac{\max_{i,j} K_b^{ij}}{\sum_i\sum_j K_o^{ij} + n}\right)\log^2\sigma \\ &\leq O(\alpha\sqrt{\sigma}\log^2\sigma)\end{aligned}$$

where the second inequality follows from the fact that i, j are each bounded by $\log\sigma$, and the last inequality follows directly from Theorem 3.1. This proves our main result, which we restate in the following theorem.

THEOREM 4.1. *Let $\mathcal{S}$ be a set of n objects in the plane, with aspect ratio bound α and scale factor bound σ. Then, $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma}\log^2\sigma)$.*

5 Extension to Higher Dimensions

The proof extends easily to d dimensions, for $d \geq 3$. The structure of the proof remains exactly the same. We tile the d-dimensional space with boxes (L_∞ balls). The main difference arises in the number of neighboring boxes for a given box B_i. While in the plane, a box has at most 5^2 neighboring boxes in the two surrounding layers, this number increases to 5^d in d dimensions. Since our arguments have been volume based, they hold in d dimensions as well. Our main theorem in d dimensions can be stated as follows.

THEOREM 5.1. *Let $\mathcal{S}$ be a set of n polyhedral objects in d-space, with aspect ratio bound α and scale factor bound σ. Then, $\rho(\mathcal{S}) = O(\alpha\sqrt{\sigma}\log^2\sigma)$, where the constant is about 5^d.*

6 Lower Bound Constructions

We first describe a construction of a family $\mathcal{S}$ with $\sigma = 1$, which shows $\rho(\mathcal{S}) = \Omega(\alpha)$. The construction works in any dimension d, but for ease of exposition, we describe it in two dimensions. See Figure 5 for illustration.

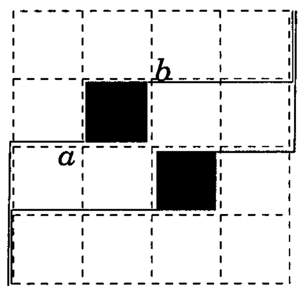

Figure 5: The lower bound construction showing $\rho(\mathcal{S}) = \Omega(\alpha)$.

Consider a square box B of size α in the standard position, namely, $B = [0, \sqrt{\alpha}] \times [0, \sqrt{\alpha}]$. We can pack roughly α unit boxes in B, in a regular grid pattern; the number is $\lfloor \sqrt{\alpha} \rfloor^2$ to be exact. We convert each of these unit boxes into a polyhedral object of aspect ratio α, by attaching two "wire" extensions at the two endpoints of its main diagonal. Specifically, consider one such unit box u, the endpoints of whose main diagonal have coordinates (a_1, a_2) and (b_1, b_2). The b endpoint of u is connected to the point $(\sqrt{\alpha}, \sqrt{\alpha})$ with a Manhattan path, whose ith edge is parallel to the *positive* i-coordinate axes and has length $\sqrt{\alpha} - b_i$. Similarly, the a endpoint of u is connected to the origin with a Manhattan path, whose ith edge is parallel to the *negative* i-coordinate axes and has length a_i. It is easy to see that each unit box, together with the two wire extensions forms a polyhedral object with aspect ratio α. By a small perturbation, we can ensure that no two objects intersect. The bounding boxes of each object pair intersect, however, and so we have at least $\binom{\alpha}{2}$ bounding box intersections in B.

We can group our n objects into $\lfloor n/\alpha \rfloor$ groups, each group corresponding to a α-size box as above. This gives us

$$K_b \geq \lfloor \frac{n}{\alpha} \rfloor \times \binom{\alpha}{2} = \Omega(n\alpha).$$

On the other hand, $K_o = 0$, and thus, $\rho(\mathcal{S}) = \Omega(n\alpha/n) = \Omega(\alpha)$.

We next generalize this construction to establish a lower bound of $\Omega(\alpha\sqrt{\sigma})$, assuming that $\alpha\sigma \leq n$. See Figure 6.

We take a square box B' of volume $4\alpha\sigma$. We divide the lower right quadrant of B' into α subboxes of size σ. We take a copy of the construction of Figure 5, scale it up by a factor of σ, and put it in place of the lower right quadrant of B'. We extend the wires attached to each

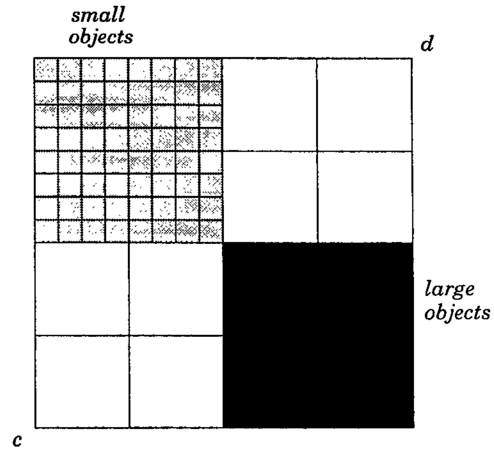

Figure 6: The lower bound construction, showing $\rho(\mathcal{S}) = \Omega(\alpha\sqrt{\sigma})$.

object to the corners c, d of B'. Thus, the smallest enclosing box of each object is now exactly B', and aspect ratio is 4α. These are the big objects. Next, we take the upper-left quadrant, divide it into σ subboxes of size α each. At each α-size subbox, we place a copy of the construction in Figure 5. These are the small objects.

Altogether we want $X = n/(1 + \sqrt{\sigma})$ big objects, and $Y = n\sqrt{\sigma}/(1 + \sqrt{\sigma})$ small objects. Since there are a total of α locations for big objects, we superimpose X/α copies of the big object at each location. Similarly, there are $\alpha\sigma$ locations for the small objects, so superimpose $Y/\alpha\sigma$ copies of the small object at each location. (This is where we need the condition $\alpha\sigma \leq n$, since we want so ensure that each location receives at least one object.) Let us now estimate bounds for K_b and K_o. The enclosing box of every big object intersects the enclosing box of every small object, we have

$$K_b \geq XY \geq \frac{n^2\sqrt{\sigma}}{(1 + \sqrt{\sigma})^2} \tag{6.4}$$

On the other hand, the only object pair intersections exist between objects assigned to the same location. We therefore have

$$\begin{aligned} K_o &\leq \alpha\binom{X/\alpha}{2} + \alpha\sigma\binom{Y/\alpha\sigma}{2} \\ &\leq \alpha(X/\alpha)^2 + \alpha\sigma(Y/\alpha\sigma)^2 \\ &\leq \frac{\sigma X^2 + Y^2}{\alpha\sigma} \\ &\leq \frac{2n^2}{\alpha(1 + \sqrt{\sigma})^2}. \end{aligned}$$

Thus,

$$\begin{aligned}\rho(\mathcal{S}) &= \frac{K_b}{K_o + n} \\ &\geq \left(\frac{n^2\sqrt{\sigma}}{(1+\sqrt{\sigma})^2}\right) / \left(\frac{n^2}{2\alpha(1+\sqrt{\sigma})^2} + n\right) \\ &\geq \frac{\alpha\sqrt{\sigma}}{2 + \frac{\alpha(1+\sqrt{\sigma})^2}{n}} \\ &\geq c\alpha\sqrt{\sigma},\end{aligned}$$

for some constant $c > 0$. (The ratio $\frac{\alpha(1+\sqrt{\sigma})^2}{n}$ is bounded by a constant, since $\alpha\sigma \leq n$.)

THEOREM 6.1. *There exists a family $\mathcal{S}$ of n polyhedral objects with aspect ratio bound α and scale factor σ such that $\rho(\mathcal{S}) = \Omega(\alpha\sqrt{\sigma})$, assuming $\alpha\sigma \leq n$.*

7 Applications and Concluding Remarks

Theorems 4.1 and 5.1 have two interesting consequences. The first is a theoretical validation of the bounding box heuristic mentioned in Section 1. In practice, the object families tend to have bounded aspect ratio and scale factor. Thus, the number of extraneous box intersections is at most a constant factor of the number of actual object-pair intersections. This result needs no assumption about the convexity of the objects.

If the aspect ratio and scale factor grow with n, our theorem indicates their impact on the efficiency of the heuristic. The degradation of the heuristic is smooth, and not abrupt. Furthermore, the result suggests that the dependence on aspect ratio and scale factor is *not symmetric*—the complexity grows linearly with α, but only as a *square root* of σ. It is common in practice to decompose complex objects into simpler parts. Our work suggests that for collision detection purposes, reducing aspect ratio may have higher payoff that reducing scale factor. It would be interesting to verify empirically how this strategy performs in practice.

The second consequence of our theorems is an output sensitive algorithm for reporting pairwise intersections among polyhedra; the bound is the strongest for *convex* polyhedra in dimensions $d = 2, 3$. We are aware of only one non-trivial result for this problem, which holds in two dimensions. Gupta et al. [9] give an $O(n^{4/3} + K_o)$ time algorithm for reporting K_o pairs of intersecting convex polygons in the plane. The problem is wide open in three and higher dimensions.

Our theorem leads to a significantly better result in two and three dimensions for small aspect and scale bounds, and nearly optimal result for *convex polyhedra*. Given n polyhedra in two or three dimensions, we can report all pairs whose *bounding boxes* intersect in time $O(n \log^{d-1} n + K_b)$ [6, 16], where K_b is the number of intersecting bounding box pairs. If the polyhedra are convex, then the narrow phase intersection test can be performed in $O(\log^{d-1} m)$ time [4], assuming that all polyhedra have been preprocessed in linear time; m is the maximum number of vertices in a polyhedron. If the convex polyhedra have aspect ratio at most α and scale factor at most σ, then by Theorem 5.1, the total running time of the algorithm is $O(n \log^{d-1} n + \alpha\sqrt{\sigma}K_o \log^2 \sigma \log^{d-1} m)$, for $d = 2, 3$. If α and σ are constants, then the running time is $O(n \log^{d-1} n + K_o \log^{d-1} m)$, which is nearly optimal.

Finally, an obvious open problem suggested by our work is to close the gap between the upper and lower bounds on $\rho(\mathcal{S})$. We believe the correct bound is $\Theta(\alpha\sqrt{\sigma})$. Our analysis is quite loose and the actual constants of proportionality are likely to be much smaller than our estimates. It would be interesting to establish better constants both theoretically and empirically.

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